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SOLUTION OF THE PROBLEM OF DYNAMIC STABILITY IN A COOPERATIVE DIFFERENTIAL GAME WITH SIDE PAYMENTS*

N.N. DANILOV

A general sufficient condition for a dynamically stable /1-3/ solution of a cooperative differential game with side payments to exist is derived, and the dynamic properties of the set of distributions and the C-core are investigated. The problem of dynamic stability of a distribution from the solution of a game is formulated. The solution of this problem is an allocation function (af) satisfying the condition of dynamic stability of the distribution (the optimal af). The notion of an initial allocation function is introduced. A method of computing the optimal initial af is developed. The method is applied to a three-person game with integral transferable payoffs, when the solution of the game is defined as the set of distributions, the C-core, and the Shapley vector.

1. Statement of the problem.

Consider an n-person differential game $\Gamma(x_0, T-t_0)$ of prescribed duration $T-t_0$,

$$\dot{x} = f(x, u_1, \ldots, u_n), \quad x \in R^m, \quad x(t_0) = x_0 \tag{1.1}$$

$$J_i(x_0, u_1, \ldots, u_n) = \int_{t_0}^{T} h_i(x(t))\,dt + H_i(x(T)) \tag{1.2}$$

Here and henceforth, unless otherwise specified, $i = 1, \ldots, n;\ t \in [t_0, T]$.

An admissible control of the player i is any function u_i Lebesgue measurable in $[t_0, T]$ which, for every t, satisfies the condition $u_i(t) \in U_i$ ($U_i \subset R^{m_i}$ is a compactum).

Regarding system (1.1) we assume that for any initial values $x_0 \in R^m$ and any combination $(u_1, \ldots, u_n)$ of admissible controls it has a unique solution $x(\cdot)$ continuable in $[t_0, T]$. In order to simplify references to system (1.1), we will denote it by $\Sigma(x_0)$.

Let $N = \{1, \ldots, n\}$ be the set of players. We assume that the rules of the game allow the formation of coalitions $S \subset N$ and that the payoffs are transferable between players /2/.

The characteristic function (cf) is the mapping $v\colon 2^N \times R^m \times R_+^1 \to R^1$ (R_+^1 is the non-negative real half-line) which associates with each coalition $S \in 2^N$ and each initial position $(x_0, T-t_0) \in R^m \times R_+^1$ a real number $v(S; x_0, T-t_0)$ equal to the payoff secured by the coalition S in the game $\Gamma(x_0, T-t_0)$ (irrespective of the actions of the players from the set $N \setminus S$).

We assume that $v(\varnothing; x_0, T-t_0) = 0$ and

$$v(N; x_0, T-t_0) = \sup_{u_1, \ldots, u_n} \sum_{i \in N} J_i(x_0, u_1, \ldots, u_n)$$

(here sup is over the direct products of the sets of admissible controls of all players). $v(N; x_0, T-t_0)$ is the maximum payoff of the coalition N in the game $\Gamma(x_0, T-t_0)$.

The vector $\xi^\circ = (\xi_1^\circ, \ldots, \xi_n^\circ)$, satisfying the conditions $\xi_i^\circ \geqslant v(i; x_0, T-t_0)$ (individual rationality), $\xi^\circ(N) = v(N; x_0, T-t_0)$ (collective rationality) is called a distribution (here and henceforth, $\xi^\circ(S) = \sum_{i \in S} \xi_i^\circ$, $S \subset N$).

We know /4/ that the vector $\xi^\circ \in R^n$ is a distribution if and only if

$$\xi_i^\circ = v(i; x_0, T-t_0) + \alpha_i^\circ \tag{1.3}$$

$$\alpha_i^\circ \geqslant 0; \quad \alpha^\circ(N) = v(N; x_0, T-t_0) - \sum_{i \in N} v(i; x_0, T-t_0) \tag{1.4}$$

The vector $\alpha^\circ = (\alpha_1^\circ, \ldots, \alpha_n^\circ)$ satisfying the conditions (1.4) will be called a side

*Prikl.Matem.Mekhan., 53, 1, 45-59, 1989 payment vector.

The triple $\Gamma_v(x_0, T-t_0) = \langle \Sigma(x_0), N, v\rangle$ will be called a cooperative differential game with side payments. The game $\Gamma_v(x_0, T-t_0)$ in which the cf is superadditive in S, i.e.,

$$v(S; x_0, T-t_0) + v(R; x_0, T-t_0) \leqslant v(R \cup S; x_0, T-t_0)$$
$$S, R \subset N: S \cap R = \varnothing \tag{1.5}$$

is called essential, We will consider only essential games.

2. Dynamically stable principles of optimality. Let $E_v(x_0, T-t_0)$ denote the set of all distributions in the game $\Gamma_v(x_0, T-t_0)$. For a superadditive cf $\alpha^\circ(N) > 0$, and we thus have $E_v(x_0, T-t_0) \neq \varnothing$.

Let W_v: $\Gamma_v(x_0, T-t_0) \to W_v(x_0, T-t_0)$ be the mapping associating with each game $\Gamma_v(x_0, T-t_0)$ a subset $W_v(x_0, T-t_0) \subset E_v(x_0, T-t_0)$, which is called optimal. The mapping W_v will be called a principle of optimality, and the set $W_v(x_0, T-t_0)$ will be called a solution of the game $\Gamma_v(x_0, T-t_0)$.

Let $x(\cdot)$ be some trajectory of the system $\Sigma(x_0)$. Embed the game $\Gamma_v(x_0, T-t_0)$ in a family (by parameter t) of similar games $\{\Gamma_v(x(t), T-t), t_0 \leqslant t \leqslant T\}$, where $\Gamma_v(x(t), T-t) = \langle \Sigma(x(t)), N, v\rangle$. By definition, $W_v(x(t), T-t) \subset E_v(x(t), T-t)$.

Any trajectory $\bar{x}(\cdot)$ of the system $\Sigma(x_0)$ such that

$$\sum_{i\in N} J_i(\bar{x}(\cdot)) = v(N; x_0, T-t_0)$$

will be called conditionally optimal. Here

$$J_i(\bar{x}(\cdot)) = \int_{t_0}^{T} h_i(\bar{x}(t))\,dt + H_i(\bar{x}(T))$$

Definition. Let $W_v(x_0, T-t_0) \neq \varnothing$. The distribution $\xi^\circ \in W_v(x_0, T-t_0)$ is called dynamically stable if there exists an n-vector function $\beta(\cdot)$ integrable in $[t_0, T]$ such that

$$\xi^\circ \in \bigcap_{t_0\leqslant t\leqslant T}\left[\int_{t_0}^{t} \beta(\tau)\, h_N(\bar{x}(\tau))\,d\tau + W_v(\bar{x}(t), T-t)\right] \tag{2.1}$$

$$h_N(\bar{x}(t)) = h_1(\bar{x}(t)) + \ldots + h_n(\bar{x}(t))$$

$$\beta_1(t) + \ldots + \beta_n(t) = 1 \tag{2.2}$$

The solution $W_v(x_0, T-t_0)$ is called dynamically stable if all the distributions $\xi^\circ \in W_v(x_0, T-t_0)$ are dynamically stable. In this case, the trajectory $\bar{x}(\cdot)$ is called optimal.

The condition (2.2) guarantees the equality

$$\sum_{i\in N} \int_{t_0}^{t} \beta_i(\tau)\, h_N(\bar{x}(\tau))\,d\tau = \int_{t_0}^{t} h_N(\bar{x}(\tau))\,d\tau$$

If together with (2.2) we have

$$\beta_i(t) \geqslant 0 \tag{2.3}$$

then $\beta_i(t)$ in (2.1) are weights. In this case, the fraction of each player in the "total payoff" $h_N(\bar{x}(t))$ is non-negative for all t.

Theorem 1. Let $h_N(\bar{x}(t)) \neq 0$. For the solution $W_v(x_0, T-t_0)$ of the game $\Gamma_v(x_0, T-t_0)$ to be dynamically stable it is sufficient that the following conditions are satisfied along the conditionally optimal trajectory $\bar{x}(\cdot)$:

1) $W_v(\bar{x}(t), T-t) \neq \varnothing$;

2) for each distribution $\xi^\circ \in W_v(x_0, T-t_0)$ there exists a function ξ^t differentiable with respect to t such that

$$\xi^t \in W_v(\bar{x}(t), T-t) \quad \text{and} \quad \xi^{t_0} = \xi^\circ.$$

Proof. Let $\xi^\circ \in W_v(x_0, T-t_0)$, and let $\xi^t \in W_v(\bar{x}(t), T-t)$ be the distribution function differentiable with respect to t such that $\xi^{t_0} = \xi^\circ$. Construct the vector function

$$\beta(t) = -[h_N(\bar{x}(t))]^{-1} \frac{d}{ds}(\xi^s)|_t \tag{2.4}$$

The function (2.4) is integrable in $[t_0, T]$ and satisfies the conditions (2.1) and (2.2). Indeed,

$$\sum_{i\in N}\beta_i(t) = -[h_N(\bar{x}(t))]^{-1}\frac{d}{ds}[v(N;\bar{x}(s),T-s)]|_t =$$
$$-[h_N(\bar{x}(t))]^{-1}\frac{d}{ds}\left[\int_s^T h_N(\bar{x}(\tau))\,d\tau + \sum_{i\in N}H_i(\bar{x}(T))\right]\Big|_t = 1$$
$$\int_t^T \beta(\tau)h_N(\bar{x}(\tau))\,d\tau + H(\bar{x}(T)) = \int_t^T d\xi^\tau + H(\bar{x}(T)) = \xi^t \tag{2.5}$$

The condition (2.1) is a direct consequence of the equality (2.5). Since the distribution $\xi^\circ \in W_v(x_0, T-t_0)$ is arbitrary, the solution $W_v(x_0, T-t_0)$ is dynamically stable.

Note that if the vector function $\beta(\cdot)$ can be chosen to be continuous in $[t_0, T]$, then the conditions of Theorem 1 are also necessary.

Corollary. In addition to the conditions of Theorem 1 let $h_N(\bar{x}(t)) > 0$ and let the function ξ^t in condition 2 be monotone non-increasing. Then for each distribution $\xi^\circ \in W_v(x_0, T-t_0)$ there exists a vector function $\beta(\cdot)$ which satisfies the conditions (2.1)-(2.3).

In some cases, the solution $W_v(x_0, T-t_0)$ is a convex, closed, polyhedral set (see below, Sects.3 and 4).

Theorem 2. For dynamic stability of the solution $W_v(x_0, T-t_0)$ which is a closed, convex, polyhedral set, it is necessary and sufficient that its extreme distributions are dynamically stable.

Proof. The necessity is obvious. Let us prove the sufficiency. Assume that $W_v(x_0, T-t_0)$ has l extreme points $\xi^{\circ,1}, \ldots \xi^{\circ,l}$. For any distribution $\xi^\circ \in W_v(x_0, T-t_0)$ there exists a single-value representation $\xi^\circ = \lambda_1\xi^{\circ,1} + \ldots + \lambda_l\xi^{\circ,l}$, where $\lambda_k \geqslant 0$, $k = 1, \ldots, l$; $\lambda_1 + \ldots + \lambda_l = 1$. By the dynamic stability of the extreme points, we obtain

$$\xi^\circ = \sum_{k=1}^{l}\lambda_k\xi^{\circ,k} = \int_{t_0}^{t}\left[\sum_{k=1}^{l}\lambda_k\beta^k(\tau)\right]h_N(\bar{x}(\tau))\,d\tau + \sum_{k=1}^{l}\lambda_k\xi^{t,k} \tag{2.6}$$
$$\xi^{t,k} \in W_v(\bar{x}(t), T-t), \quad k = 1, \ldots, l$$

Let

$$\beta(t) = \sum_{k=1}^{l}\lambda_k\beta^k(t), \quad \xi^t = \sum_{k=1}^{l}\lambda_k\xi^{t,k}$$

The vector ξ^t belongs to the set $W_v(\bar{x}(t), T-t)$ because the latter is convex, and $\beta_1(t) + \ldots + \beta_n(t) = 1$. Thus, from (2.6) we obtain that the distribution ξ° is dynamically stable.

3. The set of distributions.

In this section, the solution $W_v(x_0, T-t_0)$ of the game $\Gamma_v(x_0, T-t_0)$ is the entire set of distributions $E_v(x_0, T-t_0)$.

Lemma 1. The set of distributions $E_v(x_0, T-t_0)$ is a convex, compact subset of the space R^n with n extreme points of the form $\xi^{\circ,k} = (\xi_1^{\circ,k}, \ldots, \xi_n^{\circ,k})$, $k = 1, \ldots, n$, where

$$\xi_i^{\circ,k} = \begin{cases} v(k; x_0, T-t_0) + \alpha^\circ(N), & i = k \\ v(i; x_0, T-t_0), & i \in N \setminus \{k\} \end{cases} \tag{3.1}$$

Proof. We use the shorthand notation $E_v = E_v(x_0, T-t_0)$, $v(S; x_0, T-t_0) = v(S)$. The set E_v is convex and compact in R^n as the intersection of the hyperplane $\xi^\circ(N) = v(N)$ with the convex polyhedral domain Q: $\xi_i^\circ \geqslant v(i)$. The domain Q, being the intersection of n half-spaces in R^n, has $n(n-1)$-dimensional faces, and the point $\eta = (v(1), \ldots, v(n))$ belongs to each of these faces. The set E_v is not identical with any of the $(n-1)$-dimensional faces of Q, because otherwise the vector η would be a distribution, which is impossible in an essential game.

We will show that all vectors of the form $\xi^{\circ,k}$ are extreme points of the set E_v. Clearly, each vector $\xi^{\circ,k}$, $k = 1, \ldots, n$, is a distribution (see (1.3)-(1.4)). Assume that the vector $\xi^{\circ,k}$ is not an extreme point of the set E_v. Then there should exist two different distributions $\bar{\xi}^\circ, \bar{\bar{\xi}}^\circ \in E_v$ with components $\bar{\xi}_i^\circ = v(i) + \bar{\alpha}_i^\circ$, $\bar{\bar{\xi}}_i^\circ = v(i) + \bar{\bar{\alpha}}_i^\circ$, $i = 1, \ldots, n$, where $\bar{\alpha}_i^\circ \neq \bar{\bar{\alpha}}_i^\circ$ for at least one $i \in N$ and which satisfy the equality $\lambda\bar{\xi}^\circ + (1-\lambda)\bar{\bar{\xi}}^\circ = \xi^{\circ,k}, 0 < \lambda < 1$. Hence for the k-the components of the distributions $\bar{\xi}^\circ, \bar{\bar{\xi}}^\circ, \xi^{\circ,k}$ we should have the equality

$$\lambda\bar{\xi}_k^\circ + (1-\lambda)\bar{\bar{\xi}}_k^\circ = \xi_k^{\circ,k}, \quad 0 < \lambda < 1 \tag{3.2}$$

If $\bar{\alpha}_k^\circ = \bar{\bar{\alpha}}_k^\circ$, then from (3.2) we obtain $\bar{\alpha}_k^\circ = v(N) - \sum_{i\in N}v(i) = \bar{\bar{\alpha}}_k^\circ$. This means that $\bar{\alpha}_l^\circ = \bar{\bar{\alpha}}_l^\circ = 0$ for all $l \in N \setminus \{k\}$ and so $\bar{\xi}^\circ = \bar{\bar{\xi}}^\circ = \xi^{\circ,k}$. We should therefore have $\bar{\alpha}_k^\circ \neq \bar{\bar{\alpha}}_k^\circ$, and thus

$$\bar{\alpha}_k^\circ, \bar{\bar{\alpha}}_k^\circ < v(N) - \sum_{i\in N}v(i) \tag{3.3}$$

From (3.2) we have

$$\lambda = \frac{\xi_k^{\circ,k} - \bar{\bar{\xi}}_k^{\circ}}{\bar{\xi}_k^{\circ} - \bar{\bar{\xi}}_k^{\circ}} = \frac{v(N) - \sum_{i \in N} v(i) - \bar{\bar{\alpha}}_k^{\circ}}{\bar{\alpha}_k^{\circ} - \bar{\bar{\alpha}}_k^{\circ}}$$

If $\bar{\alpha}_k^{\circ} > \bar{\bar{\alpha}}_k^{\circ}$, then using (3.3) we obtain that the last fraction is greater than 1, which contradicts the condition $\lambda < 1$; if $\bar{\alpha}_k^{\circ} < \bar{\bar{\alpha}}_k^{\circ}$, then the last fraction is negative (because its numerator is positive), which contradicts the condition $\lambda > 0$. Thus, the assumption is false, and each of the n vectors $\xi^{\circ,k}$, $k = 1, \ldots, n$ with the components (3.1) is an extreme point of the set E_v.

We will now show that apart from the points $\xi^{\circ,k}$ $k = 1, \ldots, n$, there are no other extreme points in E_v. To this end it suffices to show that any distribution of the form $\xi^{\circ} = \{v(i) + \alpha_i^{\circ}, i = 1, \ldots, n\}$, where $\bar{\alpha}_i^{\circ} < \alpha^{\circ}(N)$ for all $i = 1, \ldots, n$ and $\alpha_i^{\circ} > 0$ for at least two indices i, is not an extreme point of the set E_v. Consider n distributions $\xi^{\circ,k}$, $k = 1, \ldots, n$ whose components are defined by the equalities (3.1) and the vector λ with the components $\lambda_i = \alpha_i^{\circ}/\alpha^{\circ}(N)$. In an essential game $\alpha^{\circ}(N) > 0$. We can verify that

$$\xi^{\circ} = \sum_{k=1}^{n} \lambda_k \xi^{\circ,k}, \quad \sum_{k=1}^{n} \lambda_k = 1, \quad 0 \leqslant \lambda_k < 1, \quad k - 1, \ldots, n$$

and for each k such that $\alpha_k^{\circ} > 0$ we have $\lambda_k > 0$. Thus, the distribution ξ° is not an extreme point of the set E_v. In order to complete the proof, it remains to note that among the vectors $\xi^{\circ,k}$, $k = 1, \ldots, n$, which are extreme points of the set E_v no two vectors are equal (otherwise the game would be inessential, see (3.1)). Thus, E_v has precisely n extreme points.

As follows from Lemma 1, any distribution $\xi^{\circ} \in E_v(x_0, T - t_0)$ is uniquely representable in the form

$$\xi^{\circ} = \xi^{\circ}(\lambda^{\circ}) = \sum_{k=1}^{n} \lambda_k^{\circ} \xi^{\circ,k} \tag{3.4}$$

where $\lambda^{\circ} = (\lambda_1^{\circ}, \ldots, \lambda_n^{\circ})$ is a vector from the n-dimensional standard simplex

$$\Lambda = \{\lambda \in R^n \mid \lambda_k \geqslant 0, k = 1, \ldots, n; \sum_{k=1}^{n} \lambda_k = 1\}.$$

In (3.4), the numbers $\lambda_1^{\circ}, \ldots, \lambda_n^{\circ}$ are the barycentric coordinates of the vector ξ° in the set $E_v(x_0, T - t_0)$. We can show that for each $\lambda^{\circ} \in \Lambda$ the vector (3.4) is a distribution. Conversely, for each distribution $\xi^{\circ} \in E_v(x_0, T - t_0)$ there exists a vector $\lambda^{\circ} \in \Lambda$, which satisfies the condition (3.4). Hence we obtain the representation

$$E_v(x_0, T - t_0) = \{\xi^{\circ}(\lambda) \in R^n \mid \lambda \in \Lambda\} \tag{3.5}$$

Consider the set-valued mapping $(x, T - t) \to E_v(x, T - t)$ which with each initial position $(x, T - t)$ associates a convex compact set of distributions in the game $\Gamma_v(x, T - t)$.

Let $\bar{x}(\cdot)$ be a conditionally optimal trajectory in the game $\Gamma_v(x_0, T - t_0)$. Define the set $X(\bar{x}(\cdot)) = \{(x, T - t) \mid x = \bar{x}(t), t_0 \leqslant t \leqslant T\}$. Let ρ_X be the Hausdorff metric induced by the metric

$$\rho(\xi^t, \eta^t) = \max_{i \in N} |\xi_i^t - \eta_i^t|, \xi^t, \eta^t \in E_v(x, T - t)$$

Lemma 2. Let the cf v be continuous on the set $X(\bar{x}(\cdot))$. Then the mapping $(x, T - t) \to E_v(x, T - t)$ is continuous by inclusion (in the metric ρ_X) on the set $X(\bar{x}(\cdot))$.

Proof. Continuity of v implies continuity of the extreme distribution functions $\xi^{t,k}$, $k = 1, \ldots, n$:

$$\xi_i^{t,k} = \begin{cases} v(k; \bar{x}(t), T - t) + \alpha^t(N), & i = k \\ v(i; \bar{x}(t), T - t), & i \in N \setminus \{k\} \end{cases}$$

Therefore, any distribution function $\xi^t = \xi^t(\lambda) = \xi(\lambda)(\bar{x}(t), T - t)$ is continuous on $X(\bar{x}(\cdot))$. This means that for each fixed $\bar{\lambda} \in \Lambda$ and any $\varepsilon > 0$ there exists $\delta_1 = \delta(\varepsilon, \bar{\lambda}) > 0$, such that $\| \xi(\bar{\lambda})(\bar{x}(t'), T - t') - \xi(\bar{\lambda})(\bar{x}(t''), T - t'')\| < \varepsilon$, whenever $\| \bar{x}(t') - \bar{x}(t'') \| < \delta_1$, $|t' - t''| < \delta_1$, $t', t'' \in [t_0, T]$. Fix some distribution $\xi^{t'} = \xi^{t'}(\bar{\lambda}) \in E_v(\bar{x}(t'), T - t')$ and take the bound

$$\inf_{\xi^{t''} \in E_v(\bar{x}(t''), T - t'')} \rho(\xi^{t'}(\bar{\lambda}), \xi^{t''}) = \inf_{\mu \in \Lambda} \rho(\xi^{t'}(\bar{\lambda}), \xi^{t''}(\mu)) \leqslant$$
$$\rho(\xi^{t'}(\bar{\lambda}), \xi^{t''}(\bar{\lambda})) < \varepsilon$$

(here we have used the representation (3.5)). Hence

$$\sup_{\xi^{t'} \in E_v(\bar{x}(t'), T - t')} \inf_{\xi^{t''} \in E_v(\bar{x}(t''), T - t'')} \rho(\xi^{t'}, \xi^{t''}) = \sup_{\lambda \in \Lambda} \inf_{\mu \in \Lambda} \rho(\xi^{t'}(\lambda), \xi^{t''}(\mu)) < \varepsilon$$

whenever

$$\|\bar{x}(t') - \bar{x}(t'')\| < \delta, \quad |t' - t''| < \delta, \quad t', t'' \in [t_0, T], \quad \delta = \min\{\delta(\varepsilon, \lambda), \lambda \in \Lambda\} \tag{3.6}$$

Similarly we obtain the bound

$$\sup_{\xi^{t''} \in E_v(\bar{x}(t''), T-t'')} \inf_{\xi^{t'} \in E_v(\bar{x}(t') T-t')} \rho(\xi^{t'}, \xi^{t''}) < \varepsilon$$

Thus, $\rho x(E_v(\bar{x}(t'), T-t'), E_v(\bar{x}(t''), T-t'')) < \varepsilon$ whenever (3.6) hold.

Let $H_N(\bar{x}(T)) = \sum_{i \in N} H_i(\bar{x}(T))$.

Lemma 3. Let $h_N(\bar{x}(t)) \neq 0$ and let the function $v(i; \bar{x}(t), T-t)$ be continuously differentiable with respect to t. Then every distribution $\xi^\circ = (\xi_1^\circ, \ldots, \xi_n^\circ) \in E_v(x_0, T-t_0)$ is representable in the form

$$\xi_i^\circ = \int_{t_0}^{T} \beta_i(t) h_N(\bar{x}(t))\, dt + H_i(\bar{x}(T)) \tag{3.7}$$

where $\beta(\cdot) = (\beta_1(\cdot), \ldots, \beta_n(\cdot))$ is a vector function integrable in $[t_0, T]$ that satisfies the condition

$$\sum_{i \in N} \beta_i(t) = 1 \tag{3.8}$$

Proof. There exists a vector $\lambda^\circ = (\lambda_1^\circ, \ldots, \lambda_n^\circ) \in \Lambda$, such that $\xi^\circ = \xi^\circ(\lambda^\circ) = \sum_{k=1}^{n} \lambda_k^\circ \xi^{\circ, k}$, where $\xi^{\circ, k}$, $k = 1, \ldots, n$, are extreme points of the set $E_v(x_0, T-t_0)$. In the interval $[t_0, T]$ construct the vector function $\beta(\cdot) = (\beta_1(\cdot), \ldots, \beta_n(\cdot))$,

$$\beta_i(t) = \lambda_i^\circ + [h_N(\bar{x}(t))]^{-1}\Big(\lambda_i^\circ \frac{d}{ds}\Big[\sum_{j \in N} v(j; \bar{x}(s), T-s)\Big]\Big|_t - \frac{d}{ds}[v(i; \bar{x}(s), T-s)]\big|_t\Big) \tag{3.9}$$

The function (3.9) are integrable in $[t_0, T]$. We will show that the vector function $\beta(\cdot)$ satisfies the condition (3.7). The integral on the right-hand side of (3.7) is

$$\lambda_i^\circ \int_{t_0}^{T} h_N(\bar{x}(t))\, dt + \lambda_i^\circ \int_{t_0}^{T} \frac{d}{ds}\Big[\sum_{i \in N} v(i; \bar{x}(s), T-s)\Big]\Big|_t dt - \int_{t_0}^{T} \frac{d}{ds}[v(i; \bar{x}(s), T-s)]\big|_t dt$$

Using the definition of a conditionally optimal trajectory, the continuity of the integrands in the second and the third integrals, and finally the additivity of the function $v(S; \bar{x}(T), 0)$ in S, we obtain that the right-hand side of the equality (3.7) can be written in the form

$$\begin{aligned}
&\lambda_i^\circ [v(N; x_0, T-t_0) - H_N(\bar{x}(T))] - \lambda_i^\circ \sum_{i \in N} v(i; x_0, T-t_0) + \\
&v(i; x_0, T-t_0) + \lambda_i^\circ \sum_{j \in N} v(j; \bar{x}(T), 0) - v(i; \bar{x}(T), 0) + H_i(\bar{x}(T)) = \\
&v(i; x_0, T-t_0) + \lambda_i^\circ \Big[v(N; x_0, T-t_0) - \sum_{i \in N} v(i; x_0, T-t_0)\Big]
\end{aligned}$$

It follows from (3.1) and (3.4) that the right-hand side of the last equality is equal to ξ_i°. Therefore, equality (3.7) holds for the vector (3.9). We can verify that the equality (3.8) also holds for the vector (3.9).

The vector function $\beta(\cdot)$ satisfying the conditions (3.7)-(3.8) will be called an allocation function (af) for the distribution ξ°.

We can show that "sufficiently many" afs exist for each distribution.

Theorem 3. Let $h_N(\bar{x}(t)) \neq 0$ and let the cf v be continuous on the set $X(\bar{x}(\cdot))$ and continuously differentiable with respect to t. Then the set of distributions $E_v(x_0, T-t_0)$ is dynamically stable.

Proof. In an essential game, $E_v(\bar{x}(t), T-t) \neq \varnothing$. The conditions of the theorem enable us to apply Lemmas 2 and 3. Therefore for each $\xi^\circ \in E_v(x_0, T-t_0)$ a continuous distribution function $\xi^t \in E_v(\bar{x}(t), T-t)$, $\xi^{t_0} = \xi^\circ$ exists. This function is differentiable. Thus, the conditions of Theorem 1 are satisfied if we take $W_v(\bar{x}(t), T-t) \subset E_v(\bar{x}(t), T-t)$.

4. C-core. In this section, the set $W_v(x_0, T-t_0)$ is the C-core $C_v(x_0, T-t_0)$ of the game $\Gamma_v(x_0, T-t_0)$. Recall that $\xi^\circ \in C_v(x_0, T-t_0)$ if and only if

$$\xi^{\circ}(S) \geqslant v(S; x_0, T - t_0), \; S \subset N \tag{4.1}$$

Let

$$\alpha^t(S) = v(S; \bar{x}(t), T-t) - \sum_{i \in S} v(i; \bar{x}(t), T-t), \quad S \subset N$$

Lemma 4. For the C-core $C_v(\bar{x}(t), T-t)$ of the game $\Gamma_v(\bar{x}(t), T-t)$ to be non-empty, it is necessary and sufficient that vectors $\lambda \in \Lambda$, exist such that

$$\sum_{i \in S} \lambda_i \geqslant \begin{cases} 0, & |S| = 1 \\ \alpha^t(S)[\alpha^t(N)]^{-1}, & |S| > 1 \end{cases} \tag{4.2}$$

(for $|S| = n$, (4.2) reduces to an equality).

Proof. Necessity. Let $C_v(\bar{x}(t), T-t) \neq \varnothing$. By (4.1), for each $\xi^t \in C_v(\bar{x}(t), T-t)$,

$$\xi^t(S) \geqslant v(S; \bar{x}(t), T-t), \; S \subset N \tag{4.3}$$

The components of the distribution ξ^t can be represented in the form

$$\xi_i^t = v(i; \bar{x}(t), T-t) + \lambda_i \alpha^t(N) \tag{4.4}$$

Substituting (4.4) into (4.3), we obtain (4.2).

Sufficiency. Consider the vector $\xi^t = (\xi_1^t, \ldots, \xi_n^t)$, whose components are given by (4.4), where λ_i satisfy condition (4.2). We have

$$\xi_i^t \geqslant v(i; \bar{x}(t), T-t); \quad \xi^t(N) = v(N; \bar{x}(t), T-t)$$
$$\xi^t(S) \geqslant v(S; \bar{x}(t), T-t), \; S \subset N$$

Therefore the vector ξ^t is a distribution in the game $\Gamma_v(\bar{x}(t), T-t)$ and belongs to its C-core $C_v(\bar{x}(t), T-t)$.

Assume that $C_v(\bar{x}(t), T-t) \neq \varnothing$ and consider the set-valued mapping $(x, T-t) \to C_v(x, T-t)$ which associates with each initial position $(x, T-t) \in X(\bar{x}(\cdot))$ a non-empty C-core $C_v(x, T-t)$ (is closed, convex set) of the game $\Gamma_v(x, T-t)$.

Lemma 5. Let $C_v(\bar{x}(t), T-t) \neq \varnothing$, and let the cf v be continuous on the set $X(\bar{x}(\cdot))$. Then the mapping $(x, T-t) \to C_v(x, T-t)$ is continuous by inclusion (in the Hausdorff metric) on the set $X(\bar{x}(\cdot))$.

Proof. We have to show that for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that

$$\rho_X(C_v(\bar{x}(t'), T-t'), C_v(\bar{x}(t), T-t)) < \varepsilon \tag{4.5}$$

whenever

$$\| \bar{x}(t') - \bar{x}(t) \| < \delta(\varepsilon), \quad |t' - t| < \delta(\varepsilon), \quad t', t \in [t_0, T] \tag{4.6}$$

The inequality (4.5) holds if and only if for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that

1) for each $\xi^t \in C_v(\bar{x}(t), T-t)$ there is $\xi^{t'} \in C_v(\bar{x}(t'), T-t')$, such that $\rho(\xi^t, \xi^{t'}) < \varepsilon$,

2) for each $\xi^{t'} \in C_v(\bar{x}(t'), T-t')$ there is $\xi^t \in C_v(\bar{x}(t), T-t)$, such that $\rho(\xi^{t'}, \xi^t) < \varepsilon$, whenever (4.6) holds.

Let us prove 1). Since the set $E_v(\bar{x}(t), T-t)$ is continuous by inclusion (Lemma 2) for every $\xi^t \in C_v(\bar{x}(t), T-t)$ and any $\varepsilon > 0$ there exists a distribution $\xi^{t'} \in E_v(\bar{x}(t'), T-t')$, such that

$$\rho(\xi^t, \xi^{t'}) < \varepsilon \tag{4.7}$$

whenever (4.6) holds. Assume that for all $\xi^{t'}$ satisfying (4.7) we have $\xi^{t'} \notin C_v(\bar{x}(t'), T-t')$. This means that

$$O_\varepsilon(\xi^t) \cap C_v(\bar{x}(t'), T-t') = \varnothing \tag{4.8}$$

where $O_\varepsilon(\xi^t) = \{\eta \mid \rho(\xi^t, \eta) < \varepsilon\}$ is an ε-neighbourhood of the distribution ξ^t. Let $\eta^{t'}$ be the orthogonal projection of the distribution ξ^t on the set $C_v(\bar{x}(t'), T-t')$,

$$\rho(\xi^t, \eta^{t'}) = \min_{\xi^{t'} \in C_v(\bar{x}(t'), T-t')} \rho(\xi^t, \xi^{t'})$$

By the compactness and convexity of the set $C_v(\bar{x}(t'), T-t')$, the point $\eta^{t'}$ exists, is unique, and lies on its boundary. Hence there follows the existence of a coalition $S^* \subset N$ such that $\eta^{t'}(S^*) = v(S^*; \bar{x}(t'), T-t')$. From (4.8) it follows that $\rho(\xi^t, \eta^{t'}) \geqslant \varepsilon$, whence we obtain the bound

$$\xi^t(S^*) \leqslant \eta^{t'}(S^*) - \varepsilon_1, \quad 0 < \varepsilon_1 \leqslant |S^*| M\varepsilon \; (M - \text{const} > 0) \tag{4.9}$$

By the continuity of the function $v(S^*; \bar{x}(t), T-t)$ in the interval $[t_0, T]$, we can choose $\varepsilon > 0$ so that

$$|v(S^*; \bar{x}(t), T-t) - v(S^*; \bar{x}(t'), T-t')| < \varepsilon_1$$

From the last inequality and the relationship (4.9) we obtain $\xi^t(S^*) < v(S^*; \bar{x}(t), T-t)$. We have a contradiction: since $\xi^t \in C_v(\bar{x}(t), T-t)$, we should have $\xi^t(S) \geqslant v(S; \bar{x}(t), T-t)$, $S \subset N$. Therefore, assumption (4.8) is false and we have proved proposition 1). Proposition 2) is proved similarly.

The following theorem is a corollary of Lemmas 3 and 4 and Theorem 1.

Theorem 4. Let $h_N(\bar{x}(t)) \neq 0$ and let the cf v be continuous on the set $X(\bar{x}(\cdot))$ and continuously differentiable with respect to t. Then for the dynamic stability of the C-core it is sufficient that condition (4.2) holds.

5. The problem of dynamic stability. Let $\xi^\circ \in E_v(x_0, T-t_0)$. Define the set (see Lemma 3)

$$B(\xi^\circ) = \Big\{\beta(\cdot) = \big(\beta_1(\cdot), \ldots, \beta_n(\cdot)\big) \,|\, \xi^\circ = \int_{t_0}^T \beta(t) h_N(\bar{x}(t))\, dt + H(\bar{x}(T)); \beta_1(t) + \ldots + \beta_n(t) = 1\Big\}$$

where $\beta(\cdot)$ is the af for the distribution ξ°. Informally, $B(\xi^\circ)$ is the set of all afs for the distribution ξ° along the conditionally optimal trajectory $\bar{x}(\cdot)$.

The final outcome of the game is always a single distribution. Assume that the players have agreed to implement the distribution $\xi^\circ \in W_v(x_0, T-t_0)$. Since this distribution is realizable only if it is dynamically stable, the players have to solve the following problem of the dynamic stability of the distribution ξ° (problem DS).

Problem DS. Find a dynamically stable realization of the distribution $\xi^\circ \in W_v(x_0, T-t_0)$ in the time interval $[t_0, T]$ along the optimal trajectory $\bar{x}(\cdot)$. In other words, from the set $B(\xi^\circ)$ select a vector function $\bar{\beta}(\cdot)$ such that

$$\Big[\int_t^T \bar{\beta}(\tau) h_N(\bar{x}(\tau))\, d\tau + H(\bar{x}(T))\Big] \in W_v(\bar{x}(t), T-t) \tag{5.1}$$

The solution $\bar{\beta}(\cdot) \in B(\xi^\circ)$ of this problem is called the optimal af for the distribution ξ°.

The solution of problem DS exists only if the distribution $\xi^\circ \in W_v(x_0, T-t_0)$ is dynamically stable.

6. Initial af. As follows from formulas (3.1) and (3.4), for each distribution $\xi^\circ = (\xi_1^\circ, \ldots, \xi_n^\circ)$ there exists a unique vector $\lambda^\circ \in \Lambda$ such that

$$\xi_i^\circ = v(i; x_0, T-t_0) + \lambda_i^\circ \alpha^\circ(N) \tag{6.1}$$

From (6.1) we obtain expressions for the barycentric coordinates $\lambda_1^\circ, \ldots, \lambda_n^\circ$ of the distribution ξ° on the set $E_v(x_0, T-t_0)$:

$$\lambda_i^\circ = [\alpha^\circ(N)]^{-1}(\xi_i^\circ - v(i; x_0, T-t_0)) \tag{6.2}$$

Construct the functions

$$\alpha_i^t = \lambda_i^\circ \alpha^t(N) \tag{6.3}$$

Clearly,

$$\xi^t = \{v(i; \bar{x}(t), T-t) + \alpha_i^t, i = 1, \ldots, n\} \in E_n(\bar{x}(t), T-t) \tag{6.4}$$

Assume that the cf v is differentiable with respect to t on $[t_0, T]$ and construct the vector function $\beta^\circ(\cdot) = (\beta_1^\circ(\cdot), \ldots, \beta_n^\circ(\cdot))$,

$$\beta_i^\circ(t) = -[h_N(\bar{x}(t))]^{-1} \frac{d}{ds}[v(i; \bar{x}(s), T-s) + \alpha_i^s]|_t \tag{6.5}$$

Applying (6.3) and (6.4) we can verify that $\beta^\circ(\cdot) \in B(\xi^\circ)$, i.e., $\beta^\circ(\cdot)$ is an af for the distribution ξ°. Noting that the vector function $\beta^\circ(\cdot)$ is computed from the barycentric coordinates $\lambda_1^\circ, \ldots, \lambda_n^\circ$ of the distribution ξ° (see (6.5), (6.3) and (6.2)) in the initial set of distributions $E_v(x_0, T-t_0)$, the af $\beta^\circ(\cdot)$ will be called the initial af for the distribution ξ°.

Let us investigate the question of the optimality of the initial af for the set of distributions, the C-core, and also the Shapley vector.

First we use the entire set of distributions as the solution of the game $\Gamma_v(x_0, T-t_0)$.

Theorem 5. For any dynamically stable distribution $\xi^\circ \in E_v(x_0, T-t_0)$, m its initial af is the optimal af (the solution of problem DS).

Proof. Consider an arbitrary distribution $\xi^\circ \in E_v(x_0, T-t_0)$. Let $\beta^\circ(\cdot) = (\beta_1^\circ(\cdot), \ldots, \beta_n^\circ(\cdot))$ be the initial af of this distribution. Integrating the equality (6.5) on $[t, T]$ and using the relationship $\alpha_i^T = 0$, $v(i; \bar{x}(T), 0) = H_i(\bar{x}(T))$, we obtain

$$I_i \triangleq \int_t^T \beta_i^\circ(\tau) h_N(\bar{x}(\tau))\, d\tau + H_i(\bar{x}(T)) = v(i; \bar{x}(t), T-t) + \alpha_i^t \tag{6.6}$$

Let $\lambda_1^\circ, \ldots, \lambda_n^\circ$ be the barycentric coordinates of the distribution ξ° in the set $E_v(x_0, T-t_0)$. Substitute in the last equality the value of α_i^t from (6.3). Since $(\lambda_1^\circ, \ldots, \lambda_n^\circ) \in \Lambda$, the vector ξ^t with the components $\xi_i^t = v(i; \bar{x}(t), T-t) + \lambda_i^\circ \alpha^t(N)$ is a distribution in the current game $\Gamma_v(\bar{x}(t), T-t)$. Thus, from (6.6) it follows that $I \in E_v(\bar{x}(t), T-t)$ $(I = (I_1, \ldots, I_n))$.

Now take the C-core as the solution of the game $\Gamma_v(x_0, T-t_0)$

Theorem 6. For the initial af $\beta^\circ(\cdot)$ of the dynamically stable distribution ξ° from the C-core $C_v(x_0, T-t_0)$ to be optimal (a solution of problem DS) it is sufficient that for each coalition $S \subset N$ the ratio $\alpha^t(S)[\alpha^t(N)]^{-1}$ is monotone non-increasing with time along the trajectory $\bar{x}(\cdot)$.

Proof. Let $\lambda_1^\circ, \ldots, \lambda_n^\circ$ be the barycentric coordinates of the distribution $\xi^\circ \in C_v(x_0, T-t_0)$. By Lemma 4,

$$\sum_{i\in S} \lambda_i^\circ \geqslant \alpha^\circ(S)[\alpha^\circ(N)]^{-1}, \quad S \subset N$$

Since $\alpha^t(S)[\alpha^t(N)]^{-1}$, is monotone non-increasing, we obtain

$$\sum_{i\in S} [v(i; \bar{x}(t), T-t) + \lambda_i^\circ \alpha^t(N)] \geqslant v(S; \bar{x}(t), T-t), \quad S \subset N$$

Therefore, at the instant t there exists a distribution

$$\xi^t = (\xi_1^t, \ldots, \xi_n^t)\colon \xi_i^t = v(i; \bar{x}(t), T-t) + \lambda_i^\circ \alpha^t(N) \tag{6.7}$$

that belongs to the C-core $C_v(\bar{x}(t), T-t)$ of the game $\Gamma_v(\bar{x}(t), T-t)$. Differentiate equality (6.7) and divide both sides of the resulting relationship by $h_N(\bar{x}(t)) \neq 0$. Using (6.3) and (6.5), we can write

$$[h_N(\bar{x}(t))]^{-1} \frac{d}{ds}(\xi_i^s)|_t = -\beta_i^\circ(t)$$

Hence we have $\xi_i^t = I_i$ (I_i is defined by equality (6.6)). Thus, $I = (I_1, \ldots, I_n) \in C_v(\bar{x}(t), T-t)$.

Theorem 7. For optimality of the initial af $\beta^\circ(\cdot)$ of the dynamically stable distribution $\xi^\circ \in C_v(x_0, T-t_0)$ it is sufficient that the barycentric coordinates of the distribution ξ° in $E_v(x_0, T-t_0)$ satisfy the condition

$$\sum_{i\in S} \lambda_i^\circ \geqslant \alpha^t(S)[\alpha^t(N)]^{-1}, \quad S \subset N$$

In other words, for optimality of $\beta^\circ(\cdot)$ for the distribution $\xi^\circ \in C_v(x_0, T-t_0)$ it is sufficient that at each instant of time t there exists a distribution $\xi^t \in C_v(\bar{x}(t), T-t)$, whose barycentric coordinates in $E_v(\bar{x}(t), T-t)$ coincide with the barycentric coordinates of the distribution ξ° in $E_v(x_0, T-t_0)$.

Theorem 7 is proved in the same way as Theorem 6. It provides a relatively simple criterion for checking the optimality of the af for the distribution ξ° from the C-core: it suffices to check the conditions (4.2) for the barycentric coordinates of ξ°.

We now give an algorithm that solves the problem DS for the case when its solution is the initial af:

1) compute the barycentric coordinates of the distribution $\xi^\circ \in W_v(x_0, T-t_0)$ in the set $E_v(x_0, T-t_0)$ (using formula (6.2));

2) find the form of the side payment functions α_i^t, $i = 1, \ldots, n$, defined in $[t_0, T]$ and corresponding to the distribution ξ° (using formula (6.3));

3) find the initial af $\beta^\circ(\cdot)$ for the distribution ξ° (using formula (6.5)).

7. Shapley vector. Let $W_v(x_0, T-t_0)$ be the Shapley vector $\Phi^v(x_0, T-t_0)$. In the current game $\Gamma_v(\bar{x}(t), T-t)$ its components are calculated from the formula

$$\Phi_i^v(\bar{x}(t), T-t) = \sum_{S\subset N: i\in S} \frac{(n-|S|)!\,(|S|-1)!}{n!} \times$$
$$[v(S; \bar{x}(t), T-t) - v(S \setminus \{i\}; \bar{x}(t), T-t)]$$

Construct the vector function $\beta(\cdot) = (\beta_1(\cdot), \ldots, \beta_n(\cdot))$,

$$\beta_i(t) = -[h_N(\bar{x}(t))]^{-1} \frac{d}{ds}[\Phi_i^v(\bar{x}(s), T-s)]|_t \tag{7.1}$$

Theorem 8. Let the vector function $\Phi^v(\bar{x}(t), T-t)$ be differentiable with respect to t. Then there exists a unique optimal af (the solution of problem DS) for the Shapley vector $\Phi^v(x_0, T-t_0)$ whose components have the form (7.1).

Proof. In each current game, the Shapley vector exists and is unique. Therefore its dynamic stability implies that

$$\Phi^v(x_0, T-t_0) = \int_{t_0}^{t} \beta(\tau) h_N(\bar{x}(\tau))\, d\tau + \Phi^v(\bar{x}(t), T-t) \tag{7.2}$$

Using the equalities

$$\sum_{i \in N} \Phi_i^v(\bar{x}(t), T-t) = v(N; \bar{x}(t), T-t), \quad \Phi^v(\bar{x}(T), 0) = H(\bar{x}(T)),$$

we obtain for the functions (7.1)

$$\sum_{i \in N} \beta_i(t) =$$
$$-[h_N(\bar{x}(t))]^{-1} \frac{d}{ds}\left[\int_s^T h_N(\bar{x}(\tau))\, d\tau + H_N(\bar{x}(T))\right]|_t = 1$$
$$\int_{t_0}^{T} \beta_i(t) h_N(\bar{x}(t))\, dt + H_i(\bar{x}(T)) =$$
$$-\int_{t_0}^{T} d\Phi_i^v(\bar{x}(t), T-t) + H_i(\bar{x}(T)) = \Phi_i^v(x_0, T-t_0)$$

Therefore, the vector function $\beta(\cdot)$ with the components (7.1) is an af for $\Phi^v(x_0, T-t_0)$. The vector function $\beta(\cdot)$ is a solution of problem DS for $\Phi^v(x_0, T-t_0)$, because it reduces equality (7.2) to an identity. The uniqueness of the optimal af follows from the uniqueness of the Shapley vector in each current game.

8. Solution of problem DS for one three-person game. Consider the three-person differential game described by the equation

$$\begin{aligned} &\dot{x} = u_1 + u_2 + u_3, \quad x(t_0) = 0, \quad t_0 = 0 \\ &x = (x^{(1)}, x^{(2)}), \quad u_i = (u_i^{(1)}, u_i^{(2)}) : \|u_i\| \leqslant 1, \quad i = 1, 2, 3 \end{aligned} \tag{8.1}$$

The payoff functions have the form (a_i, b_i, c_i are non-negative constants)

$$\begin{aligned} &J_i(x_0, u_1, u_2, u_3) = \int_{t_0}^{T} (a_i x^{(1)}(t) + b_i x^{(2)}(t) + c_i)\, dt \\ &\left(\sum_{i \in S} a_i\right)^2 + \left(\sum_{i \in S} b_i\right)^2 \neq 0, \quad S \subset N \quad (N = \{1, 2, 3\}) \end{aligned} \tag{8.2}$$

Admissible controls are the functions u_i Lebesgue measurable in $[t_0, T]$ such that at each instant $t \in [t_0, T]$ $\|u_i(t)\| \leqslant 1$.

Let us construct the cf v. To this end, for each coalition $S \subset N$ consider the zero-sum game $\Gamma_S^y(x_0, T-t_0)$ between the coalitions S and $N \setminus S$, which is defined as follows. The game dynamics is (8.1). We use piecewise-programmed strategies (PPS) /2/ as the admissible strategies φ_S and $\varphi_{N \setminus S}$ of the players S and $N \setminus S$ in the game Γ_S^y. The set of PPS of the player S ($N \setminus S$) is denoted by D_S ($D_{N \setminus S}$). The payoff of the player S (the maximizing player) in each situation $(\varphi_S, \varphi_{N \setminus S})$ is defined as $-\theta(x(T), y)$, where $x(T)$ is the final point of the trajectory $x(\cdot) = x(\cdot; x_0, \varphi_S, \varphi_{N \setminus S})$ of the system (8.1), $y \in R^2$, and θ is the Euclidean metric. Since the right-hand side of Eq. (8.1) is "separable" in the controls and the integrands in (8.2) are continuous in R^2, then for each $\varepsilon > 0$ there exists and ε-saddle point $(\varphi_S^\varepsilon, \varphi_{N \setminus S}^\varepsilon)$ and the value

$$\operatorname{val} \Gamma_S^y(x_0, T-t_0) = \lim_{\varepsilon \to 0} J(x_0, \varphi_S^\varepsilon, \varphi_{N \setminus S}^\varepsilon)$$
$$(J(x_0, \varphi_S^\varepsilon, \varphi_{N \setminus S}^\varepsilon) = -\theta(x(T; x_0, \varphi_S^\varepsilon, \varphi_{N \setminus S}^\varepsilon), y)),$$

For each coalition $S \subset N$ define the set

$$Y_S^{T-t_0}(x_0) = \{y \in R^2 \mid \operatorname{val} \Gamma_S^y(x_0, T-t_0) = 0\}$$

Let

$$r_S(t_0) = (|S| - |N \setminus S|)(T - t_0), \quad R(x_0, r_S(t_0)) = \{x \in R^2 \mid \|x\|^2 \leqslant r_S^2(t_0)\}$$

Lemma 6. Let $r_S(t_0) > 0$. For each PPS $\varphi_{N\setminus S} \in D_{N\setminus S}$ and each point $y \in R(x_0, r_S(t_0))$ there exists a PPS $\varphi_S \in D_S$, attaining the point y (a point y is attained in the situation $(\varphi_S, \varphi_{N\setminus S})$ if $x(T; x_0, \varphi_S, \varphi_{N\setminus S}) = y$).

Proof. Assume that the player $N\setminus S$ chooses the PPS $\varphi_{N\setminus S} = (\Delta_{N\setminus S}, a_{N\setminus S})$. Construct the PPS $\varphi_S = (\Delta_S, a_S)$, where $\Delta_S = \{t_0 = t_0^{\Delta_S} < \ldots < t_{l_S}^{\Delta_S} = T\}$, $a_S = (a_S^0, \ldots, a_S^{l_S})$, $a_S^k: (t_k^{\Delta_S}, x(t_k^{\Delta_S})) \to u_S$, $\|u_S\| \leqslant |S|$, $k = 0, \ldots, l_S$, in the following way: $\Delta_S = \Delta_{N\setminus S}$ and

$$a_S^k(x) = -a_{N\setminus S}^k(x) + \omega, \quad \forall x \in R^2, \quad k = 0, \ldots, l_S \tag{8.3}$$

$$\omega = \frac{y - x_0}{T - t_0}, \quad y \in R(x_0, r_S(t_0))$$

Since

$$\|\omega\| - \frac{1}{T - t_0}\|y - x_0\| \leqslant \frac{1}{T - t_0} r_S(t_0) = |S| - |N\setminus S|$$

then

$$\|a_S^k(x)\| = \|a_{N\setminus S}^k(x)\| + \|\omega\| \leqslant |N\setminus S| + |S| - |N\setminus S| = |S|$$

and the relationship (8.3) is thus well-defined.

For the pair of PPS $(\varphi_S, \varphi_{N\setminus S})$, we obtain from (8.1) $\dot{x} = \omega$, or

$$x(T) = x_0 + \frac{1}{T - t_0}\int_{t_0}^{T} (y - x_0)\, dt = y$$

Corollary. Let $r_S(t_0) > 0$. Then

$$Y_S^{T-t_0}(x_0) = \{x \in R^2 \mid \|x\|^2 \leqslant r_S^2(t_0)\} \neq \varnothing \tag{8.4}$$

Proof. By Lemma 6,

$$\inf_{\varphi_S} \sup_{\varphi_{N\setminus S}} \theta(x(T; x_0, \varphi_S, \varphi_{N\setminus S}), y) = 0, \quad \forall y \in R(x_0, r_S(t_0))$$

Since the right-hand sides of (8.1) are separable in controls, we may interchange **inf** and **sup**, so that

$$\operatorname{val} \Gamma_S^y(x_0, T - t_0) = 0, \quad \forall y \in R(x_0, r_S(t_0))$$
$$\operatorname{val} \Gamma_S^y(x_0, T - t_0) > 0, \quad \forall y \notin R(x_0, r_S(t_0))$$

Therefore (8.4) holds.

Since the inequality $r_S(t_0) < 0$ is equivalent to the inequality $r_{N\setminus S}(t_0) > 0$, then (8.4) remains valid when S is replaced by $N\setminus S$ and $r_S(t_0)$ by $r_{N\setminus S}(t_0)$.

Now define the cf

$$v(S; x_0, T - t_0) = \begin{cases} \max \sum_{i \in S} J_i, & |S| > |N\setminus S| \\ \min \sum_{i \in S} J_i, & |S| < |N\setminus S| \\ 0, & S = \varnothing \end{cases} \tag{8.5}$$

where J_i is the payoff function (8.2), the operations max and min are over $Y_S^{T-t_0}(x_0)$ and $Y_{N\setminus S}^{T-t_0}(x_0)$, respectively.

For each coalition $S \subset N$, let

$$u_S = (u_S^{(1)}, u_S^{(2)}), \quad u_S^{(m)} = \sum_{i \in S} u_i^{(m)}, \quad m = 1, 2$$
$$a_S = \sum_{i \in S} a_i, \quad b_S = \sum_{i \in S} b_i, \quad c_S = \sum_{i \in S} c_i$$

Compute the conditionally optimal trajectory $\bar{x}(\cdot) = x(\cdot; x_0, \bar{u}_N)$:

$$\sum_{i \in N} J_i(x_0, \bar{u}_N) = \max_{u_N} \sum_{i \in N} J_i(x_0, u_N)$$

Using Pontryagin's maximum principle, we obtain

$$\bar{u}_N^{(1)} = \frac{3a_N}{\sqrt{a_N^2 + b_N^2}}, \quad \bar{u}_N^{(2)} = \frac{3b_N}{\sqrt{a_N^2 + b_N^2}}$$

Hence

$$x^{(1)}(t) = \frac{3a_N}{\sqrt{a_N^2 + b_N^2}}(t - t_0) + x_0^{(1)}$$
$$x^{(2)}(t) = \frac{3b_N}{\sqrt{a_N^2 + b_N^2}}(t - t_0) + x_0^{(2)}$$

For the maximum coalition N we have

$$v(N; x(t), T - t) = \frac{3}{2}\sqrt{a_N^2 + b_N^2}(T^2 - t^2) + [(a_N x_0^{(1)} + b_N x_0^{(2)} + c_N) - 3\sqrt{a_N^2 + b_N^2}\, t_0](T - t)$$

Similarly we find

$$\bar{u}_S^{(1)} = \frac{a_S}{\sqrt{a_S^2 + b_S^2}}, \quad \bar{u}_S^{(2)} = \frac{b_S}{\sqrt{a_S^2 + b_S^2}}, \quad S \subset N: |S| = 2$$
$$\bar{u}_i^{(1)} = -\frac{a_i}{\sqrt{a_i^2 + b_i^2}}, \quad \bar{u}_i^{(2)} = -\frac{b_i}{\sqrt{a_i^2 + b_i^2}}$$
$$v(S; x(t), T - t) = \frac{1}{2}\sqrt{a_S^2 + b_S^2}(T - t)^2 + \sum_{i \in S}(a_i x^{(1)}(t) + b_i x^{(2)}(t) + c_i)(T - t), S \subset N: |S| = 2; \quad v(i; x(t), T - t) = -\frac{1}{2}\sqrt{a_i^2 + b_i^2}(T - t)^2 + (a_i x^{(1)}(t) + b_i x^{(2)}(t) + c_i)(T - t)$$

Lemma 7. The cf v defined by Eq.(8.5) is superadditive in S.

The lemma is proved by substituting the values of the cf v into (1.5).

Thus, on the basis of the game (8.1), (8.2) we have defined an essential cooperative game $\Gamma_v(x_0, T - t_0)$ in the form of the cf (8.5).

Since the conditions of Theorem 3 are satisfied, the set of distributions $E_v(x_0, T - t_0)$ in this game is dynamically stable.

For a three-person game, the condition (4.2) for the C-core $C_v(x(t), T - t)$ to exist is equivalent to the inequality

$$\sum_{S \subset N: |S| = 2} v(S; x(t), T - t) \leqslant 2v(N; x(t), T - t)$$

This condition is obviously true in our case, and so $C_v(x(t), T - t) \neq \varnothing$. All the conditions of Theorem 4 are satisfied, and therefore the C-core $C_v(x_0, T - t_0)$ in this game exists and is dynamically stable.

Thus, if the solution $W_v(x_0, T - t_0)$ of the game $\Gamma_v(x_0, T - t_0)$ is defined as the entire set of distributions or the C-core, the conditionally optimal trajectory $\bar{x}(\cdot)$ is optimal.

Applying the algorithm of Sect.6, we will solve problem DS for the set of distributions, the C-core, and the Shapley vector.

In (8.2) let $T = 1$; $a_1 = 1$, $a_2 = 2$, $a_3 = 0$; $b_1 = 2$, $b_2 = b_3 = 1$; $c_1 = 10$, $c_2 = 15$, $c_3 = 5$. For these numerical values, we calculate (to two decimal places)

$$v^\circ(N) = 37.50, \quad v^\circ(1) = 8.88, \; v^\circ(2) = 13.88, \; v^\circ(3) = 4.50$$
$$v^\circ(\{1,2\}) = 27.12, \quad v^\circ(\{1,3\}) = 16.58, \quad v^\circ(\{2,3\}) = 21.41$$

where $v^\circ(S) = v(S; x_0, T - t_0)$. At an arbitrary instant $t \in [0, 1]$ we have

$$h_N(\bar{x}(t)) = \sum_{i=1}^{3}(a_i \bar{x}^{(1)}(t) + b_i \bar{x}^{(2)}(t) + c_i) = 15t + 30$$
$$v^t(N) = 7.50(1 - t^2) + 30(1 - t); \; v^t(1) = -7.72t^2 - 1.16t + 8.88$$
$$v^t(2) = -7.12t^2 - 6.76t + 13.88, \; v^t(3) = -2.90t^2 - 1.60t + 4.50$$

$$v^t(\{1,2\}) = -10.48t^2 - 16.64t + 27.12$$
$$v^t(\{1,3\}) = -7.42t^2 - 9.16t + 16.58$$
$$v^t(\{2,3\}) = -6.99t^2 - 14.42t + 21.41$$

where $v^t(S) = v(S; \bar{x}(t), T-t)$.

Let us solve problem DS for the distribution

$$\xi^\circ = (10;\ 20;\ 7.50) \in E_v(x_0, T-t_0) \tag{8.6}$$

Compute the barycentric coordinates of the distribution (see (6.2))

$$\lambda_1^\circ = \frac{1.12}{10.24},\quad \lambda_2^\circ = \frac{6.12}{10.24},\quad \lambda_3^\circ = \frac{3}{10.24}$$

Find the side payment functions α_i^t, $i = 1, 2, 3$ (using formula (6.3)) corresponding to the distribution ξ° (i.e., for $t = 0$ we should have $\xi_i^\circ = v(i; x_0, T-t_0) + \alpha_i^\circ$, $i = 1, 2, 3$)

$$\alpha_1^t = \lambda_1^\circ [v(N;\ \bar{x}(t),\ T-t) - \sum_{i=1}^{3} v(i; \bar{x}(t), T-t)] = 1.12t^2 - 2.24t + 1.12$$

Similarly, $\alpha_2^t = 6.12t^2 - 12.24t + 6.12$, $\alpha_3^t = 3t^2 - 6t + 3$.

Find the initial af (see (6.5))

$$\beta_1^\circ(t) = -[h_N(\bar{x}(t))]^{-1} \frac{d}{ds}[v(i;\ \bar{x}(s),\ T-s) + \alpha_i^s]|_t = \frac{13.20t + 3.40}{15t + 30}$$

Similarly,

$$\beta_2^\circ(t) = \frac{2t + 19}{15t + 30},\quad \beta_3^\circ(t) = \frac{-0.20t + 7.60}{15t + 30}$$

Note that the vector $\beta^\circ(t) = (\beta_1^\circ(t), \beta_2^\circ(t), \beta_3^\circ(t))$ satisfies conditions (2.2) and (2.3), i.e., for each $t \in [0, 1]$ its components are weights.

By Theorem 5 the vector function $\beta^\circ(\cdot)$ is a solution of problem DS for the distribution ξ°. Indeed, we can verify that

$$\begin{aligned}\xi^t &= \int_t^1 \beta^\circ(\tau)\, h_N(\bar{x}(\tau))\, d\tau = (-6.60t^2 - 3.40t + 10,\\ &-t^2 - 19t - 20,\ 0.10t^2 - 7.60t + 7.50) = \{v(i; \bar{x}(t), 1-t) + \alpha_i^t,\ i = 1, 2, 3\}\end{aligned} \tag{8.7}$$

and so

$$\int_t^1 \beta^\circ(\tau)\, h_N(\bar{x}(\tau))\, d\tau \in E_v(\bar{x}(t),\ 1-t),\quad 0 \leqslant t \leqslant 1$$

Using $\beta^\circ(\cdot)$ we can compute the dynamically stable payoffs earned by the players in any time interval $[0, t]$ given the distribution ξ°. For instance, in $[0, {}^1/_2]$ these payoffs are

$$\int_0^{1/2} \beta^\circ(\tau)\, h_N(\bar{x}(\tau))\, d\tau = (3.35,\ 9.75,\ 3.77)$$

The vector of the remaining payoffs in the time interval $[{}^1/_2, 1]$, i.e., (6.65, 10.25, 3.73), belongs to the set $E_v(\bar{x}({}^1/_2), {}^1/_2)$, i.e., it is optimal in the same sense as the distribution ξ°.

Let us solve problem DS for the distribution $\xi^\circ \in C_v(x_0, T-t_0)$. Let us investigate the behaviour of the ratio $\gamma_S(t) = \alpha^t(S)\,[\alpha^t(N)]^{-1}$ (see Theorem 6) for increasing $t \in [0, 1]$. For S: $|S| = 1, \gamma_S(t) \equiv 0$ for all $t \in [0, 1]$, and for $S = N, \gamma_S(t) \equiv 1$ for all $t \in [0, 1]$. Let $|S| = 2$. We have

$$\alpha^t(N) = 10.24\,(t-1)^2;\quad \alpha^t(\{1.2\}) = 4.36\,(t-1)^2,$$
$$\alpha^t(\{1.3\}) = 3.20\,(t-1)^2,\quad \alpha^t(\{2.3\}) = 3.03\,(t-1)^2$$

We see that the ratio $\gamma_S(t)$ for each coalition is constant along the optimal trajectory. By Theorem 6, the solution of problem DS for $\xi^\circ \in C_v(x_0, T-t_0)$ is its initial af.

We will show this for the distribution (8.6) (it can be checked that this distribution belongs to the C-core $C_v(x_0, T-t_0)$). We have to show that for any $t \in [0, 1]$ the vector (8.7) belongs to the current C-core. By Theorem 7, it suffices to check condition (4.2) for the barycentric coordinates of the distribution ξ°. It can be shown that these conditions indeed hold. Therefore, the af $\beta^\circ(\cdot)$ is indeed optimal for ξ°.

Consider the Shapley vector. Since the functions $\Phi_i^v(\bar{x}(t), T-t)$, $i = 1, 2, 3$ are differentiable with respect to t, the functions (7.1) exist, and therefore the vector $\Phi^v(x_0, T-t_0)$ is dynamically stable (see (7.2)). The trajectory $\bar{x}(\cdot)$ is thus an optimal trajectory for the Shapley vector.

The current Shapley vector has the form

$$\begin{aligned}\Phi^v(\bar{x}(t),\ T-t) &= \{\Phi_i^v(\bar{x}(t),\ T-t),\ i = 1, 2, 3\} = (-4.06t^2 -\\ &8.48t + 12.54,\ -3.54t^2 - 13.92t + 17.46,\ 0.10t^2 - 7.60t + 7.50)\end{aligned}$$

By formula (7.1), we obtain the optimal af $\bar{\beta}(\cdot)$ for the Shapley vector $\Phi^v(x_0, T-t_0)$:

$$\bar{\beta}(t) = \left(\frac{8.12t+8.48}{15t+30}, \frac{7.08t+13.92}{15t+30}, \frac{-0.20t+7.60}{15t+30}\right), \quad 0 \leqslant t \leqslant 1$$

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A DIFFERENTIAL GAME WITH A FUZZY TARGET SET AND FUZZY STARTING POSITIONS*

V.A. BAIDOSOV

A mathematical model of a situation in which it is required to develop a single control strategy for a fuzzy set of objects in the presence of noise is considered. The control objective is to hit a fuzzy target set at a given instant of time or to evade the target set. The problem reduces to constructing a universal optimal strategy for a differential game whose payoff function is the membership function of the target set.

1. Consider the differential game

$$\begin{gathered} \mathbf{x}^{\cdot} = f(t, \mathbf{x}, u, v) \\ \mathbf{x} \in R^n, \quad u(t) \in P, \quad v(t) \in Q \end{gathered} \tag{1.1}$$

where P and Q are compacta in R^p and R^q. We assume that the right-hand side of (1.1) satisfies the canonical conditions /1, p.37, 38/ and that the small-game saddle-point condition holds /1, p.79/. The game is considered in the time interval $T \overset{\Delta}{=} [t_*, \theta]$.

Let u: $T \times R^n \to P$ be some positional strategy of the first player. We denote by $K_u[t_0, t](\mathbf{x}_0)$ the set of constructive motions /2, p.33/ $\mathbf{x}(\cdot)$ generated by the strategy u in the time interval $[t_0, t]$ and satisfying the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$. Let the set X be the set of all non-empty subsets of the space X. We define the set-valued mapping

$$K_u(t, t_0)\colon R^n \to \operatorname{set} R^n, \quad t \geqslant t_0$$

setting

$$K_u(t, t_0)\mathbf{x}_0 \overset{\Delta}{=} \{\mathbf{x}(t) : \mathbf{x}(\cdot) \in K_u[t_0, t](\mathbf{x}_0)\}$$

We similarly define the set-valued mappings $K_v(t, t_0)$ for the second-player strategy v.

Let $F(X)$ be the family of fuzzy sets in the space X, μ_A the membership function of a fuzzy set. The value of μ_A at the point $\mathbf{x}$ will be denoted by $\langle \mu_A, \mathbf{x} \rangle$.

If some mapping

$$\pi\colon R^n \to F(R^n) \tag{1.2}$$

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